MATH 4010 | 26/02/2015

Tutorial 6

Eg 1. (Adjoint Operator on finite dimentional spaces.)

Let T be a linear operator from n-dimentional vector space X to r-dimentional vector space Y. Then T is represented by a $r \times n$ matrix A, and the adjoint operator $T^{\times}$ is represented by A^{T}.

<u>Remark 1</u>: A linear operator on finite dimentional spaces is bounded.

<u>Remark 2</u>: Adjoint Operator can be regard as the generalization of the transpose of matrix.

<u>Pf</u>: (i) We choose a basis $E = \{e_1, e_2, \cdots, e_n\}$ of X

and a basis $B = \{b_1, b_2, \cdots, b_r\}$ of Y

Then, $\forall x \in X$ can be uniquely represented as $x = \sum_{k=1}^{n} \xi_k e_k$.

Since T is linear, $y = Tx = \sum_{k=1}^{n} \xi_k Te_k$. That is, T is uniquely determined by $y_k := Te_k$.

Set $y = \sum_{j=1}^{r} \eta_j b_j$, $y_k := Te_k = \sum_{j=1}^{r} \tau_{jk} b_j$, since $\{b_1, \cdots, b_r\}$ is a basis

Thus, $y = \sum_{j=1}^{r} \eta_j b_j = \sum_{k=1}^{n} \xi_k \left(\sum_{j=1}^{r} \tau_{jk} b_j \right) = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \tau_{jk} \xi_k \right) b_j$

So, $\eta_j = \sum_{k=1}^{n} \tau_{jk} \xi_k$, for $j = 1, 2, \cdots, r$.

Set $A = (\tau_{jk})_{r \times n}$. Then $\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$.

Hence, T is represented by the matrix A.

(ii) Let $F = \{f_1, f_2, \cdots, f_r\}$ be the dual basis of B, i.e. $f_i(b_j) = \delta_{ij}$, $i, j = 1, 2, \cdots, r$.

Then, $\forall g \in Y'$, g can be represented by $g = \sum_{i=1}^{r} \alpha_i f_i$

So, $\forall y = Tx$, as show in (i) $y = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix}$ i.e. $\eta_j = \sum_{k=1}^{n} \tau_{jk} \xi_k$.

$$g(y) = \sum_{i=1}^{r} \alpha_i f_i(y) = \sum_{i=1}^{r} \alpha_i f_i\left(\sum_{j=1}^{r} \eta_j b_j \right) = \sum_{i=1}^{r} \alpha_i \left(\sum_{j=1}^{n} \eta_j f_i(b_j) \right)$$

$$= \sum_{i=1}^{r} \alpha_i \left(\sum_{j=1}^{r} \eta_j \delta_{ij} \right) = \sum_{i=1}^{r} \alpha_i \eta_i \left(= \sum_{j=1}^{r} \alpha_j \eta_j \right) = \sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_j \tau_{jk} \xi_k.$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{r} \tau_{jk} \alpha_j \xi_k = \sum_{k=1}^{n} \beta_k \xi_k \quad \text{with } \beta_k = \sum_{j=1}^{r} \tau_{jk} \alpha_j, \ k = 1, 2, \cdots, n$$

Define f a functional on X as

$$f(x) = g(Tx) = \sum_{k=1}^{n} \beta_k \xi_k, \text{ with } \beta_k = \sum_{j=1}^{r} \tau_{kj} \alpha_j.$$

It is clear that $f \in X'$. So, $f = T^{\times} g$ and f is represented by $\beta_k = \sum_{j=1}^{r} \tau_{kj} \alpha_j$, i.e. $T^{\times}$ is represented by $A^T = (\tau_{kj})_{n \times r}$.

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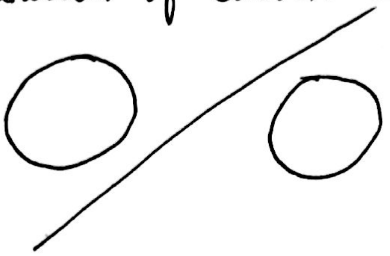
Eg 2. (Separation of convex sets.)



Let X be a normed space over $\mathbb{R}$, and let A, B be nonempty, disjoint convex subsets of X.

(i) If A is open, then $\exists$ a bounded linear fcnal $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$ s.t. $\varphi(a) < c \leq \varphi(b)$, $\forall a \in A, b \in B$

(ii) If A is compact and B is closed, then $\exists$ a bounded linear fcnal $\phi : X \to \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$ s.t. $\varphi(a) \leq c_1 < c_2 \leq \varphi(b)$, $\forall a \in A, b \in B$.

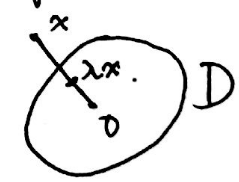
Remark The hyperplane $H_c = \{x \in X \mid \varphi(x) = c\}$ seperates the two convex sets A and B.

Pf: Choose points $a_0 \in A$, $b_0 \in B$ and set $x_0 = b_0 - a_0$.

Consider $D = A - B + x_0 = \{a - a_0 + b_0 - b \mid a \in A, b \in B\}$

Since A, B are convex sets and A is open. It is clear that D is an open convex neighborhood of 0.

Moreover, $x_0 \notin D$, otherwise, $x_0 = a - b + x_0$, i.e. $a - b = 0$ for some $a \in A$, $b \in B$, which is a contradiction to $A \cap B = \phi$.



Define $p(x) = \inf\{\lambda > 0 \mid x \in \lambda D\}$

Since D is open and $0 \in D$, $B(0, \rho) \subset D$ for some $\rho > 0$.

Thus $p(x) \leq \frac{\|x\|}{\rho}$, since $x \in \frac{\|x\|}{\rho} B(0, \rho) \subset \frac{\|x\|}{\rho} D$, $\forall x \in X$.

Furthermore, p satisfy $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ $\forall \alpha \geq 0$.

Indeed, $\forall \varepsilon > 0$, let $\lambda_1 = p(x) + \frac{\varepsilon}{2}$, $\lambda_2 = p(y) + \frac{\varepsilon}{2}$, then

$$\frac{x}{\lambda_1} \in D \text{ and } \frac{y}{\lambda_2} \in D$$

So, $\frac{x+y}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{x}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{y}{\lambda_2} \in D$ since D is convex,

that is $p(x+y) \leq \lambda_1 + \lambda_2 = p(x) + p(y) + \varepsilon \Rightarrow p(x+y) \leq p(x) + p(y)$.

Note that $x_0 \notin D$. One has $p(x_0) \geq 1$.

Set $Z = \{\alpha x_0 \mid \alpha \in \mathbb{R}\}$. Then Z is a subspace of X.

Define a functional f on Z as $f(\alpha x_0) = \alpha$.

Then $f(x_0) = 1$ and $f(\alpha x_0) = \alpha \leq \alpha p(x_0) = p(\alpha x_0)$

By the Hahn-Banach Theorem, $\exists$ a linear fcnal $\varphi : X \to \mathbb{R}$ s.t.

$$\varphi(x) \leq p(x) \text{ and } \varphi(x_0) = f(x_0) = 1.$$

Since $p(x) \leq \frac{\|x\|}{\rho}$, φ is bounded and $\|\varphi\| \leq \frac{1}{\rho}$.

Therefore, $\forall a \in A$, $b \in B$,

$$\varphi(a) - \varphi(b) + 1 = \varphi(a - b + x_0) \leq p(a - b + x_0) < 1,$$ since $a - b + x_0 \in D$, D is open.

i.e. $\varphi(a) < \varphi(b)$, $\forall a \in A$, $b \in B$.

The sets $\varphi(A)$ and $\varphi(B)$ are nonempty, disjoint convex sets and $\varphi(A)$ is open. Taking $c = \sup_{a \in A} \varphi(a)$, then (i) is proved.

(ii) Since A is compact and B is closed,

$$d(A, B) = \inf \{\|a - b\| \mid \forall a \in A, b \in B\} > 0.$$

Let $r = d(A, B)$. Then $A_r := \{x \in X : d(x, A) < r\}$ does not intersect B.

Then, (i) yields that $\exists$ a linear bounded fcnal $\varphi : X \to \mathbb{R}$ and $c_2 > 0$ s.t.

$$\varphi(x) < c_2 \leq \varphi(y) \quad \forall x \in A_r \text{ and } y \in B.$$

Since φ is cts, and A is compact, $\varphi(A)$ is compact.

So $c_1 = \sup_{x \in A} \varphi(x) < c_2$. This proves (ii)

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